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EQUILIBRIUM AND STABILITY OF AN INCOMPRESSIBLE FLUID DROP

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Equilibrium shapes and stability of axially symmetric drops were investigated in detail in [1-3]. The papers [4-6] were devoted to conditions of drop breakup following slow growth. Based on the results of [6], a mechanism was suggested [7] for determining the surface-tension coefficient of the fluid by the height of the drop at the moment of break-up. Below we consider equilibrium and stability of an axially symmetric drop, adjacent to a bulk incompressible fluid and bounded by a planar free surface. Unlike [1-7], in studying the stability of this system [8] it is necessary to take into account perturbations varying the volume of the drop. Therefore the class of stable equilibrium shapes is narrowed down.

1. Let some volume Q of an incompressible fluid be in a uniform field of mass forces and be confined by solid walls of a container S and by free surfaces Σ and Σ_1. The surface Σ_1 is planar, and Σ confines that part of the volume, protruding in the form of a drop at the outer surface of the container walls. We assume that the wall surface near the contour L is the base of the drop and the wetting characteristic of this part of the wall is axially symmetric, while the symmetry axis is parallel to the direction of the gravity field **g**. We assume that the drop is also axially symmetric. The contour radius of the drop base is denoted by R_0, and the drop height by H. We introduce a cylindrical coordinate system $\{r, z, \theta\}$ with origin at the center of the drop base and a Z axis along the symmetry axis inside the volume Q (Fig. 1). The coordinates r and z are dimensionless: $r = R/R_0$, $z = Z/R_0$.

We denote by s the path length measured from the plane of the drop along the meridian. The meridian coordinates are given parametrically

$$r = r(s),\ z = z(s),\ 0 \leqslant s \leqslant l,$$

where l is the total length of the drop meridian.

The functions r(s), z(s) satisfy the well-known [1] system of ordinary differential equations

$$r''(s) = -z'(s)q(s,\ \beta,\ \eta),\ z''(s) = r'(s)q(s,\ \beta,\ \eta),$$
$$q(s,\ \beta,\ \eta) = \beta z(s) + \eta - z'(s)/r(s) \tag{1.1}$$

and the boundary conditions (−h is the ordinate of the pole of the drop)

$$r(0) = 0,\ z(0) = -h,\ r(l) = 1,\ z(l) = 0. \tag{1.2}$$

The dimensionless parameters β, η of the system (1.1) are

$$\beta = \rho g R_0^2/\sigma, \quad \eta = p_0 R_0/\sigma, \tag{1.3}$$

where ρ is the fluid density, g is the acceleration projection of the gravity force **g** on the Z axis, σ is the surface-tension coefficient of the drop, and p_0 is the pressure at the base plane of the drop ($z \equiv 0$).

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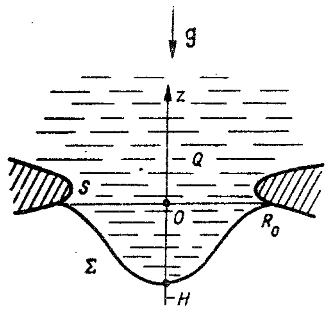

Fig. 1

For fixed values of h and η the determination of drop equilibrium shapes reduces to searching parameter values $\beta=\beta(h, \eta)$, for which the corresponding triplet of numbers $\{h, \eta, \beta\}$ obtained is an integral curve of system (1.1), starting from the point $(0, -h)$ of the plane (r, z) without leaving the half-plane $\{z \le 0\}$, and coinciding at the point $(1, 0)$. The calculation of the parameter β is carried out by an iteration method, at each step of which the system (1.1) is integrated numerically in the parameter s from $s=0$ to $s=l$, so that $z(l)=0$. The iteration leads (with some given accuracy) to the equality $r(l)=1$.

It must be noted that the boundary condition $r(0)=0$ (1.2) creates some difficulties in a numerical solution of Eq. (1.1) near the point $s=0$, since r(s) appears in the denominator of the function $q(s, \beta, \eta)$. By direct expansion of the indicated features of problem (1.1), (1.2) we find the following conditions of finiteness of solution at $s=0$:

$$r'(0) = 1,\ r''(0) = z'(0) = 0,\ z''(0) = (\eta - \beta h)/2. \tag{1.4}$$

Relations (1.4) make it possible to perform the first step at the point $s=0$ by numerical integration of system (1.1) by the Runge-Kutta method without resorting to power expansions [1].

We also note that for arbitrary h the parameters

$$\beta = 0,\ \eta = 4h/(1 + h^2) \tag{1.5}$$

correspond to the following exact solution of problem (1.1), (1.2):

$$r(s) = \frac{1+h^2}{2h}\sin\frac{2h}{1+h^2}s, \tag{1.6}$$

$$z(s) = \frac{1-h^2}{2h} - \frac{1+h^2}{2h}\cos\frac{2h}{1+h^2}s, \quad 0 \leqslant s \leqslant \frac{1+h^2}{2h}\arccos\frac{1-h^2}{1+h^2}.$$

This can be verified by direct substitution of the function (1.6) with account of (1.5) into Eqs. (1.1) and boundary conditions (1.2). The solution (1.6) determines an arc of radius $(1+h^2)/2h$, forming a spherical segment whose shape the drop would acquire under conditions of zero gravity.

For given $h>0$ and values of η differing from (1.5), the parameter β corresponding to the solution of problem (1.1), (1.2) is determined by the iteration scheme suggested above. The dependence $\beta(h, \eta)$ of the parameter β on η was calculated numerically on a computer for various fixed values of the parameter h. Figure 2 shows the curves of $\beta(h, \eta)$ for the values $h=0.1, 0.5, 1, 2.5$ (the lines 1-4, respectively). The functions $\beta(h, \eta)$ increase monotonically with η, passing through zero, according to (1.5), at $\eta=4h/(1+h^2)$. In the region $0<\eta<2$ the dependence $\beta=\beta(h, \eta)$ is nearly linear for fixed h. The line $\beta=\beta(0, \eta)$ corresponding to $h=0$ is, obviously, the β-axis, since for arbitrary β and $\eta=0$ the necessary conditions (1.1) for an equilibrium planar surface $\{z\equiv 0\}$ are satisfied identically.

The behavior of integral curves of equations of type (1.1) and the corresponding drop shapes were investigated in detail and illustrated in [1-3]. Not providing the shapes of the drop meridians, which are fully considered in [1-3], we note that for a given choice of dimensionless units drops with smaller η are, for arbitrary $h>0$, fully contained inside drops with larger values of the parameter η.

2. To determine the stability of a given drop equilibrium shape we use the approach of [8]. We assume that the container walls S near the contour L_1 of the free surface Σ_1 are vertical, and the corresponding edge angle is 90°, i.e., the surface Σ_1 is planar. Let (x, y) be Cartesian coordinates in the Σ_1 plane. The problem (1.6) of [8] in the perturbation u(x, y) of the surface Σ_1 acquires then the form

$$-\sigma_1\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \lambda u(x, y), \quad (x, y) \in \Sigma_1, \left.\frac{\partial u}{\partial n}\right|_{L_1} = 0, \tag{2.1}$$

where σ_1 is the surface tension coefficient of Σ_1, and n is the normal to the contour L_1 in the Σ_1 plane.

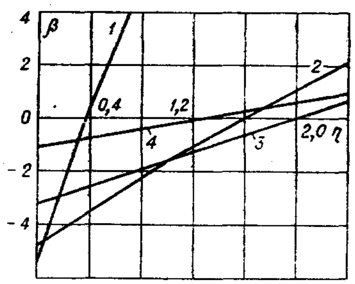

Fig. 2

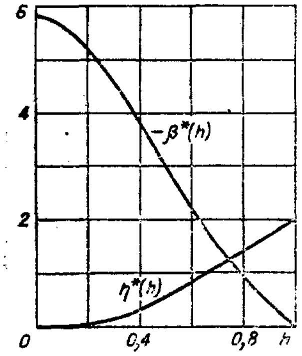

Fig. 3.

Obviously, the function $u_0(x, y) \equiv 1$ is a solution of problem (2.1), corresponding to the eigenvalue $\lambda_0 = 0$. Since the operator of problem (2.1) is self-adjoint, it follows [9] that the remaining eigenfuctions $u_k(x, y)$ ($k = 1, 2, \ldots$) of this problem satisfy the condition

$$\int_{\Sigma_1} u_0 u_k d\Sigma_1 = \int_{\Sigma_1} u_k d\Sigma_1 = 0. \tag{2.2}$$

According to [9], however, the operator of problem (2.1) with isoperimetric bound (2.2) is positively definite. Consequently, $\lambda_0 = 0$ is the smallest eigenvalue of problem (2.1).

We denote by $\varkappa_1, \varkappa_2, \ldots$ the ordered sequence of eigenvalues of problem (3.3) of [8] on the surface of the drop for the case of axially symmetric ($n = 0$) perturbations $u(s)$

$$-u''(s) - r'(s)u'(s)/r(s) + \tau(s)u(s) = \lambda u(s),\ 0 < s < l,\ u'(0) = 0,\ \chi u(l) + u'(l) = 0, \tag{2.3}$$

where $\tau(s)$ is of the form [1]

$$\tau(s) = \beta r'(s) - [z'(s)/r(s)]^2 - r''(s)^2 - r''(s)^2, \tag{2.4}$$

$r(s)$ and $z(s)$ are the solution of problem (1.1), (1.2), and the parameter χ is determined by the wettability condition of the drop contour [1, 2].

Let w_1 be the volume of the first eigenfunction of problem (2.3). Using Eq. (2.5) of [8] for $w_1 \neq 0$ or case B [8] for $w_1 = 0$, and taking into account that the smallest eigenvalue of problem (2.1) vanishes, it can be shown that the sign of the minimum second variation of the potential energy of system $\{\Sigma, \Sigma_1\}$ coincides with the sign of the first eigenvalue $\varkappa_1$ of problem (2.3). We note that since problem (2.3) for axially symmetric perturbations of the surface Σ does not have isoperimetric restrictions, it follows from inequality (8.19) of [2] that non-axially-symmetric perturbations cannot be considered. Therefore the answer to the drop stability problem is given by the sign only of the first eigenvalue $\varkappa_1$ of problem (2.3).

Consider the case in which the fluid wets totally the wall surface S up to the drop contour L. The boundary condition of problem (2.3) at the point $s = l$ acquires then the form

$$u(l) = 0. \tag{2.5}$$

In the case of problem (2.3) with restriction (2.5) for the solution of the stability problem of a given equilibrium shape it is not necessary to calculate the first eigenvalue $\varkappa_1$ of problem (2.3). It is sufficient to integrate the differential equation of problem (2.3) for $\lambda = 0$ with initial conditions

$$u(0) = 1,\ u'(0) = 0.$$

If, then, $u(s) > 0$ for all $0 < s \leq l$, the equilibrium shape is stable, while if a point $s^* \in (0, l)$ is found such that $u(s^*) = 0$, the given equilibrium shape is unstable. The stability limit is reached if $u(s) > 0$, $0 < s < l$, but $u(l) = 0$. These statements follow from the comparison theorem [10] for solutions of second-order linear differential equations.

Using the stability criterion mentioned, drop equilibrium shapes were studied for negative Bond numbers β (1.3). For various h values calculations were performed along the curves $\beta(n, \eta)$ considered in Sec. 1. Figure 3 shows the curves $\beta^*(h)$ and $\eta^*(h)$, determining the stability limit of a drop of height h (= const). For $\beta < \beta^*(h)$ the equilibrium shapes of a drop of given height h are unstable, while for $\beta > \beta(h)$, $h \in (0,1)$, they are stable. For $h > 1$ there are no stable equilibrium shapes with negative β numbers. Similar statements concerning stability for $\eta < \eta^*(h)$ or $\eta > \eta^*(h)$, respectively, follow from the monotonicity of the curves $\beta(h, \eta)$ (see Fig. 2).

We also note that $\beta^*(0) = -\xi_1^2$, where ξ_1 is the first root of the Bessel function $J_0(\xi)$ [10]. This can be shown by passing to the limits $r' \to 1$, r'', z', $z'' \to 0$ in expression (2.4) and Eq. (2.3).

TABLE 1

γ	$\alpha=-\gamma$		$\alpha^{-1}=-\gamma$		γ	$\alpha=-\gamma$		$\alpha^{-1}=-\gamma$	
	η	h	η	h		η	h	η	h
0,05	1,91748	0,97450	0,21476	0,33524	0,55	1,49774	0,81792	1,01298	0,65188
0,10	1,85316	0,95255	0,36460	0,42362	0,60	1,46901	0,80704	1,05280	0,66411
0,15	1,79851	0,93285	0,48322	0,47830	0,65	1,44160	0,79675	1,08921	0,67546
0,20	1,75011	0,91482	0,58234	0,51675	0,70	1,41531	0,78698	1,12265	0,68606
0,25	1,70618	0,89814	0,66769	0,54622	0,75	1,39004	0,77768	1,15347	0,69598
0,30	1,66571	0,88259	0,74244	0.57017	0,80	1,36572	0,76883	1,18196	0,70529
0,35	1,62804	0,86804	0,80872	0,59051	0,85	1,34235	0,76042	1,20830	0,71402
0,40	1,59275	0,85440	0,86790	0,60830	0,90	1,31976	0,75237	1,23286	0,72228
0,45	1,55937	0,84152	0,92113	0,62420	0,95	1,29807	0,74473	1,25569	0,73005
0,50	1,52778	0,82939	0,96929	0,63865	1,00	1,27703	0,73740	1,27703	0,73740

The calculations performed make it possible to formulate stability criteria of the system of equilibrium surfaces $\{\Sigma, \Sigma_1\}$ and for the case $\beta=\text{const}$. We fix some number $\beta_0 \in (-\xi_1^2, 0)$ and put $\eta_0=\eta^*(h_0)$, where h_0 is the root of the equation $\beta^*(h)=\beta_0$. For a given value of the Bond number β_0 any drop equilibrium shape with dimensionless pressure η is then primarily stable for $\eta<\eta_0$ and unstable for $\eta>\eta_0$. At the same time the drop height h satisfies the inequality $h<h_0$ or $h>h_0$, respectively.

We stress that, as shown above, in passing through a critical value the system $\{\Sigma, \Sigma_1\}$ loses stability of strict axial symmetry. Therefore the problem of possibility of transition of a drop into a near non-axially-symmetric equilibrium shape, considered in [1, 4] in studying evolution of an isolated drop, does not arise in the given case. Consequently, loss of stability under the condition $\beta=\text{const}$ is accompanied by breakup of part of the drop, since for $\eta>\eta_0$ there are no stable axially symmetric shapes.

3. As an example of applying the stability conditions obtained we consider a possible scheme of measuring the surface-tension coefficient. For some pressure p at the center of the drop base with volume Q (Sec. 1), restricted by free surfaces Σ and Σ_1 (see Fig. 1), let the drop be in a state of stable equilibrium. The pressure p can be measured quite accurately, since it is determined by the distance between the drop base and the planar surface Σ_1. If the apparatus on which the drop contour is fixed is quasistatically displaced vertically downward, the pressure p in the drop base will increase. We assume that the area of the surface Σ_1 and the volume of the fluid underneath it exceed significantly the area of the drop base and its volume. After achieving some critical pressure p_0, part of the drop is then torn, but the level of the surface Σ_1 is practically unchanged, and therefore the break-up pressure p_0 can also be measured with the required accuracy.

The experiment described is performed at a fixed value of the parameter β_0. Using expression (1.3), we form the new dimensionless parameter

$$\alpha = \beta_0/\eta_0 = \rho g R_0/p_0. \tag{3.1}$$

The parameter α (3.1) is independent of the surface-tension coefficient σ and is calculated directly from the experimental results. Since (Sec. 2) $\beta_0=\beta^*(h_0)$, $\eta_0=\eta^*(h_0)$, it follows from Fig. 3 that the parameter $\alpha(<0)$ increases monotonically on the interval $0<h\leq 1$. Therefore, along with the dependences $\beta^*(h)$, $\eta^*(h)$ one can construct the functions $\beta^*(\alpha)$, $\eta^*(\alpha)$, $-\infty<\alpha<0$, also determining the stability limit. Having, then, a graphical or tabular dependence of $\eta^*(\alpha)$, by means of (1.3), (3.1) one can then find the coefficient

$$\sigma = p_0 R_0/\eta^*(\alpha).$$

Table 1 provides the quantities η, h, corresponding to some values of the parameters $|\alpha|$ and $|\alpha|^{-1}$.

We note that the method described in [7] also leads to a series of results of determining the surface-tension coefficient by means of an approach based on the stability conditions of axially symmetric incompressible fluid drops obtained in [6]. The stability conditions of [6] were found within an exact statement of the problem, and are equivalent to the stability conditions of an isolated axially symmetric drop, derived in [1, 2]. By the method of [7] it is necessary to measure the drop height at the moment directly preceding its breakup for quasistatic extrusion from the aperture of a known radius.

A quite complete list of references is also given in [7], related to measurements of the surface-tension coefficient by static methods. The authors of [7] believe that the advantages of their approach over the earlier methods are, in particular, the following: the use of exact stability conditions relating the parameters of the problem with drop heights measured in the experiment and with the unknown coefficient, the possibility of measuring the coefficient σ by the given method at the boundary between two fluids, and the possibility of measuring σ for a high-temperature fluid, when the drop height can be measured by the same optical methods, but from a larger distance.

The experimental scheme considered above, based on the stability conditions obtained in Sec. 2 within an exact formulation, possesses the same range of applicability as the method of [7]. We only note in addition that the critical pressure in the scheme suggested can, in principle, be measured more reliably than drop height at the moment preceding breakup.

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CHARACTERISTICS IN THE INITIAL STAGE OF THE SPREADING OF A DROP ON A SOLID SURFACE

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A study of the process of the spreading of a drop on a solid surface has been the subject of many investigations (see, e.g., [1, 2]). In all of these investigations the process of spreading was considered from the instant of time when the drop could already be regarded as a liquid body, having the form of a spherical segment with an angle of wetting on the order of 90°. However, for a precise formulation of the problem concerning the spreading of a spherical drop it is necessary to have some idea of how the boundary of wetting behaves from the instant the drop makes contact with the solid surface. This is the problem we address in the present paper.

To study the initial stage in the spreading of a spherical drop on a solid plane surface we employed the experimental setup shown in Fig. 1. The principle involved here is the following. When air is admitted into the pipette 1 a spherical drop 2 is formed; upon separating from the pipette, the drop acquires the requisite speed u_0 and falls onto the plane surface 3. But before striking the surface it intersects a light ray in the optical system consisting of the light source 4 and the photocell 5; consequently, after a requisite time delay, the device 10 energizes the high-voltage RC-oscillator 9, which furnishes a series of high-voltage pulses to the hydrogen flashtube 6. Periodic flashes of light from the latter pass through the shadowgraph 7 in whose field of view the drop appears, spreads out on the solid surface, a recording of which is made by the photoregister 8 (a transparent rotating drum with a film). Thus, the process to be studied is recorded frame by frame. Moreover, with the aid of the two mirrors 11 (see Fig. 1a), a record is obtained of the spreading of the drop 2 on the transparent plate surface 3, in two projections simultaneously: from the side (rays $a-a$) and from below (rays b–b).

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